

A GENERALIZATION OF THE STRONG CASTELNUOVO LEMMA

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ABSTRACT. We consider a set X of distinct points in the n -dimensional projective space over an algebraically closed field k . Let A denote the coordinate ring of X , and let $a_i(X) = \dim_k[\mathrm{Tor}_i^R(A, k)]_{i+1}$. Green's Strong Castelnuovo Lemma (SCL) shows that if the points are in general position, then $a_{n-1}(X) \neq 0$ if and only if the points are on a rational normal curve. Cavaliere, Rossi and Valla conjectured in [2] that if the points are not necessarily in general position the possible extension of the SCL should be the following: $a_{n-1}(X) \neq 0$ if and only if either the points are on a rational normal curve or in the union of two linear subspaces whose dimensions add up to n . In this work we prove the conjecture.

1. INTRODUCTION

Let k be an algebraically closed field, and let $X = \{P_1, \dots, P_s\}$ be a set of $s \geq n+1$ distinct points in $\mathbb{P}^n := \mathbb{P}_k^n$, not contained in any hyperplane.

Let $I = I(X)$ denote the defining ideal of X in the polynomial ring $R = k[x_0, \dots, x_n]$, and let $A = R/I$ denote its homogeneous coordinate ring.

The graded R -module A has a minimal free resolution

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_1 \longrightarrow R \longrightarrow A \longrightarrow 0,$$

where $F_i = \bigoplus_{j=1}^{\beta_i} R(-d_{ij})$.

Many authors have been interested in the relation between the numerical invariants of the resolution and the geometric properties of X .

We are mostly interested in the “linear part” of the resolution, that is, the syzygies that are determined by linear forms. This study has been initiated by Green [8] and the main idea coming from his work is that “a long linear strand in the resolution has a uniform and simple motivation”. See for example [2], [3], [4], [5], [6], [7], [9], [10], [11], [12] (this is by no means a complete list) and the literature cited there.

For every $i = 1, \dots, n$, let $a_i := a_i(X) = \dim_k[\mathrm{Tor}_i^R(A, k)]_{i+1}$ denote the multiplicity of the shift $i+1$ in F_i .

This work was supported (in part) by a grant from The City University of New York PSC-CUNY Research Award Program-39.

The author would like to thank D. Eisenbud, M.E. Rossi and G. Valla for stimulating conversations about the topic of this paper.

It is well known that if $a_i = 0$ for some i , then $a_j = 0$ for all $j \geq i$. Since $a_1 = \dim_k(I_2)$, where I_2 denotes the homogeneous part of degree 2 of I , we are interested in varieties lying on some quadric.

We say that X is in general position if $n+1$ points of X are never on a hyperplane.

A well celebrated result of Green, the Strong Castelnuovo Lemma (SCL for short), shows that for a set of distinct points in \mathbb{P}^n in general position, we have that $a_{n-1} \neq 0$ (that is, there is a linear strand of length $n-1$ in the resolution) if and only if the points are on a rational normal curve of \mathbb{P}^n (see [8, 3.c.6]).

It is natural to ask what happens if the points are not necessarily in general position. Cavaliere, Rossi, and Valla conjectured in [2] that the possible extension of the SCL should be the following.

Conjecture 1.1. *For a set X of distinct points spanning \mathbb{P}^n , one has $a_{n-1} \neq 0$ if and only if either the points are on a rational normal curve or on $\mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k+r=n$.*

It follows from [2, 1.2] that if the points are on a rational normal curve or on $\mathbb{P}^k \cup \mathbb{P}^r$ with $k+r=n$, then $a_{n-1} \neq 0$. In view of this result and of the SCL, Conjecture 1.1 can be restated as follows.

Conjecture 1.2. *If X is not in general position and $a_{n-1} \neq 0$, then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r with $k+r=n$.*

In this work we prove the following theorem, which appears in Section 4 as Theorem 4.1.

Theorem 1.3. *Let X be a set of distinct points spanning \mathbb{P}^n . Fix $i = 0, \dots, n-2$. Assume that:*

- (1) *There exist $n-i+1$ points of X on a \mathbb{P}^{n-i-1} ,*
- (2) *$n-i$ points of X are never on a \mathbb{P}^{n-i-2} ,*
- (3) *$a_{n-1} \neq 0$.*

Then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k+r=n$.

Notice that if the points are not in general position, then (1) is satisfied for $i=0$. Since (2) is satisfied for $i=n-2$, Theorem 1.3 proves Conjecture 1.2.

Cavaliere, Rossi, and Valla proved Theorem 1.3 for $i=0$ and $i=1$ (cases they were interested in for other purposes, see [2, 4.2]).

Following the philosophy of [2], the main idea of this work is to study explicitly the quadrics passing through the points. We show that there are enough quadrics that “split” into the product of two linear forms to guarantee that X is contained in the union of two linear subspaces whose dimensions add up to n .

Now we briefly describe the content of this paper. In Section 2 we recall very useful tools from [2]. The main point is that $a_{n-1} \neq 0$ implies that there is at least one

nonzero quadric of the form $F_{abc} = \lambda_{abc}x_ax_b + \mu_{abc}x_ax_c + \nu_{abc}x_bx_c$, $0 \leq a < b < c \leq n$, passing through the points. We will refer to such quadrics as “special quadrics”. Remark 2.2 shows that these special quadrics are “nicely related”.

The bulk of the paper is given by Section 3. We prove a general result (Theorem 3.1) showing that if we know that certain special quadrics are reducible, then we can explicitly construct more reducible quadrics passing through the points.

In Section 4 we start the proof of Theorem 1.3. The assumptions guarantee that all quadrics F_{abj} , with $\{a, b\} \subset \{0, \dots, n-i-1\}$ and $j \in \{n-i, \dots, n\}$ “split”, $F_{abj} = x_j L_{ab}^j$, where L_{ab}^j is a linear form in x_a and x_b .

Let W_j be the vector space generated by the linear forms L_{ab}^j . First we show that if $W_j = 0$ for all $j = n-i, \dots, n$, then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k+r = n$. This statement follows easily from Theorem 3.1.

In Section 5 we complete the proof of Theorem 1.3 by proving that if $W_j \neq 0$ for some j , then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k+r = n$. We first prove the statement when $\dim W_j \geq n-i-1$ (Theorem 5.4). When $\dim W_j < n-i-1$ we use Theorem 3.1 as a starting point.

2. PRELIMINARIES

In this section we introduce the necessary notation and we recall tools that are very useful in the proof of Theorem 1.3.

We compute $\text{Tor}_i^R(A, k)$ using a resolution of the field k which can be obtained from the Koszul complex of x_0, \dots, x_n . We fix a k -vector space V of dimension $n+1$. Then the Koszul resolution of k is given by

$$0 \rightarrow \wedge^{n+1} V \otimes R(-n-1) \rightarrow \wedge^n V \otimes R(-n) \rightarrow \dots \rightarrow \wedge V \otimes R(-1) \rightarrow R \rightarrow k \rightarrow 0.$$

Let $\delta_i : \wedge^i V \otimes R(-i) \rightarrow \wedge^{i-1} V \otimes R(-i+1)$ be the usual Koszul map. We denote by K_{n-2} the kernel of δ_{n-2} in degree n . A special case of [1, 1] gives that

$$a_{n-1} = \dim_k \left[\left(\wedge^{n-2} V \otimes I_2 \right) \cap K_{n-2} \right],$$

where I_2 denotes the homogeneous part of degree 2 of the ideal I .

Let e_0, \dots, e_n be a k -vector basis of V . If j is a $(n-2)$ -tuple $\{0 \leq j_1 < \dots < j_{n-2} \leq n\}$, let $\epsilon_j := e_{j_1} \wedge \dots \wedge e_{j_{n-2}} \in \wedge^{n-2} V$. The following observations play a crucial role.

Remark 2.1. ([2, 1.3]) We have that every element $\alpha \in (\wedge^{n-2} V \otimes I_2) \cap K_{n-2}$ can be written as $\alpha = \sum_{|j|=n-2} \epsilon_j \otimes F_{C_j}$, where $C_j := \{0, \dots, n\} \setminus \{j\}$ and $F_{C_j} \in I_2$ is a square free quadratic form in the variables x_l , $l \in C_j$. \square

Therefore if $a_{n-1} \neq 0$ there is at least one nonzero quadric of the form

$$F_{abc} = \lambda_{abc}x_ax_b + \mu_{abc}x_ax_c + \nu_{abc}x_bx_c,$$

$0 \leq a < b < c \leq n$, passing through the points.

Remark 2.2. ([2, 1.4]) For every $\{a, b, c, d\}$ such that $0 \leq a < b < c < d \leq n$ we have that

$$(-1)^a x_a F_{bcd} + (-1)^{b-1} x_b F_{acd} + (-1)^{c-2} x_c F_{abd} + (-1)^{d-3} x_d F_{abc} = 0.$$

□

3. REDUCIBLE QUADRICS THROUGH THE POINTS

In this section we prove Theorem 3.1. The proof gives an explicit description of certain quadrics passing through the points that “split” into the product of two linear forms. Most of the proof of Theorem 1.3 will follow from Theorem 3.1.

Consider the quadrics F_{C_j} as in Section 2.

Theorem 3.1. *Let $j \in \{0, \dots, n\}$, and let $\{i_1, \dots, i_m\} \subset \{0, \dots, n\} \setminus \{j\}$ with $i_1 < \dots < i_m$. Suppose that $F_{efj} \in (x_j)$ for all $\{e, f | e \neq f\} \subset \{i_1, \dots, i_m\}$. Write $F_{efj} = x_j L_{ef}$, and let V be the vector space spanned by the linear forms L_{ef} . Let $d := \dim V$ and suppose that $0 < d < m - 1$. Then there exist t , with $0 < t \leq d$, linear forms L_1, \dots, L_t , which are part of a basis of V , and linearly independent linear forms h_1, \dots, h_{m-1-t} , such that*

$$(L_1, \dots, L_t)(h_1, \dots, h_{m-1-t}) \subset I.$$

Proof. Let \mathcal{L} be the set of linear forms $\{L_{ef} | e \neq f\}$. For simplicity of notation we rename the m variables involved in \mathcal{L} as y_1, \dots, y_m . If $L_{ef} \in \mathcal{L}$ and $e < f$, let

$$L_{ef} = \lambda_{ef} y_e + \mu_{ef} y_f.$$

We may assume that either $j < i_1$, or that $j > i_m$. Applying Remark 2.2 to $\{j, e, f, g\}$ (or $\{e, f, g, j\}$) with $1 \leq e < f < g \leq m$ we obtain that

$$(1) \quad F_{efg} = (-1)^{e+j} y_e L_{fg} + (-1)^{f+j-1} y_f L_{eg} + (-1)^{g+j} y_g L_{ef}.$$

The following lemma will be used often.

Lemma 3.2. *Let $L_{ef} \in \mathcal{L}$, and suppose that the coefficient of y_f in L_{ef} is not zero. Let $\{u, v\} \subset \{1, \dots, m\} \setminus \{e, f\}$, and let T be a linear form in y_u and y_v . Assume that $y_e T \in I$, and that L_{eu} and L_{ev} are monomials in y_e . Then $y_f T \in I$.*

Proof. Suppose that $y_f T \notin I$. Since $y_e T \in I$ there exists a point E such that $y_f(E) \neq 0$, $T(E) \neq 0$ and $y_e(E) = 0$. Without loss of generality assume that $y_u(E) \neq 0$, and that $e < f < u$. By (1) $F_{efu} = \pm y_e L_{fu} \pm y_f L_{eu} \pm y_u L_{ef}$. Now $F_{efu}(E) = 0$ implies that $\mu_{ef} y_f(E) y_u(E) = 0$, a contradiction. □

Suppose that at least one among the linear forms in \mathcal{L} is a non zero monomial, say L_{ab} is a monomial in y_a . If the coefficient of y_c in L_{ac} is not zero we say that y_c is *connected* to y_a (in one step).

We construct inductively a *block of monomials* B^{ab} starting with L_{ab} in the following way. At step 1 we add all the monomials connected to y_a . At step $i \geq 2$ we add new monomials connected to the monomials introduced in step $i - 1$. In other words, B^{ab} consists of all monomials connected to y_a in a finite number of steps. Notice that the set B^{ab} is part of a basis of V , and therefore it contains at most $m - 2$ monomials.

In what follows B^{ab} denotes the block of monomials starting with $L_{ab} = \lambda_{ab}y_a \neq 0$. We say that y_a is a *generator* of B^{ab} .

Remark 3.3.

- (1) If $L_{ab} = \lambda_{ab}y_a \neq 0$ and $L_{ac} = \lambda_{ac}y_a \neq 0$, then $B^{ab} = B^{ac}$.
- (2) If $y_b \in B^{ab}$ and $L_{bc} = \lambda_{bc}y_b \neq 0$, then $B^{ab} = B^{bc}$, since y_a is connected to y_b .

□

Next we describe some quadrics that factor into the product of two linear forms.

For simplicity of notation let $L_{ab} = L_{12} = \lambda_{12}y_1 \neq 0$. By (1) we have that for $s = 3, \dots, m$,

$$F_{12s} = (-1)^{j+1}y_1L_{2s} + (-1)^{j+1}y_2L_{1s} + (-1)^{s+j}y_sL_{12}.$$

If $\mu_{1s} = 0$ (which is the case if $y_s \notin B^{12}$), then

$$F_{12s} = (-1)^jy_1[(-1)^s\lambda_{12} - \mu_{2s})y_s - (\lambda_{1s} + \lambda_{2s})y_2] = (-1)^jy_1f_s,$$

where

$$(2) \quad f_s = ((-1)^s\lambda_{12} - \mu_{2s})y_s - (\lambda_{1s} + \lambda_{2s})y_2.$$

More generally, applying (1) to $1 < u < v$ we have that

$$F_{1uv} = (-1)^{j+1}y_1L_{uv} + (-1)^{u+j-1}y_uL_{1v} + (-1)^{v+j}y_vL_{1u}.$$

If $\mu_{1u} = \mu_{1v} = 0$ (which is the case if $y_u, y_v \notin B^{12}$), then

$$F_{1uv} = (-1)^jy_1[(-1)^{u-1}\lambda_{1v} - \lambda_{uv})y_u + ((-1)^v\lambda_{1u} - \mu_{uv})y_v] = (-1)^jy_1G_{uv},$$

where

$$(3) \quad G_{uv} = ((-1)^{u-1}\lambda_{1v} - \lambda_{uv})y_u + ((-1)^v\lambda_{1u} - \mu_{uv})y_v.$$

In particular, $G_{2v} = f_v$.

Remark 3.4. Let $1 < u < v < w$. By (1) and (3) we obtain that

$$\begin{aligned} F_{uvw} = & (-1)^{w+j-1}y_w[G_{uv} + (-1)^u(\lambda_{1v} + (-1)^{w-1}\mu_{vw})y_u + (-1)^{v-1}(\lambda_{1u} + (-1)^{w-1}\mu_{uw})y_v] \\ & + (-1)^{v+j-1}\lambda_{uw}y_u y_v + (-1)^{u+j}\lambda_{vw}y_u y_v. \end{aligned}$$

In particular, if $\lambda_{uw} = \lambda_{vw} = 0$ we have that $F_{uvw} = (-1)^{w+j-1}y_wG_{uv}$ if and only if $\mu_{uw} = (-1)^w\lambda_{1u}$ and $\mu_{vw} = (-1)^w\lambda_{1v}$ if and only if the coefficient of y_w in G_{uv} and in G_{vw} is zero. □

Remark 3.5. Assume that $y_2 \in B^{12}$. Let $Y_C = \{y_1, \dots, y_m\} \setminus \{B^{12}\}$ and let C be the set of indexes of the variables in Y_C . Notice that $Y_C \neq \emptyset$, since $\dim V < m - 1$. If $s \in C$, then $\mu_{1s} = \mu_{2s} = 0$. Therefore we have that $y_1 f_s \in I$, where $f_s = (-1)^s \lambda_{12} y_s - (\lambda_{1s} + \lambda_{2s}) y_2$. If $\lambda_{1s} = \lambda_{2s} = 0$ for all $s \in C$, then by Lemma 3.2

$$(4) \quad (B^{12})(Y_C) \subset I,$$

and so the conclusion of Theorem 3.1 holds.

If $\lambda_{1s} \neq 0$ for some $s \in C$, then $L_{1s} = \lambda_{1s} y_1 \neq 0$ and $B^{12} = B^{1s}$. Notice that $y_s \notin B^{1s}$. If $\lambda_{2s} \neq 0$ for some $s \in C$, then $L_{2s} = \lambda_{2s} y_2 \neq 0$ and $B^{12} = B^{2s}$, since $y_2 \in B^{12}$. Notice that $y_s \notin B^{2s}$.

Therefore we may assume that $y_2 \notin B^{12}$. \square

Let $s \neq 2$, and suppose that $y_2, y_s \notin B^{12}$. Then by Lemma 3.2

$$(5) \quad (B^{12})f_s \subset I,$$

since for every $y_e \in B^{12}$, L_{2e} and L_{es} are monomials in y_e .

More generally, if $y_u, y_v \notin B^{12}$ we have that

$$(6) \quad (B^{12})G_{uv} \subset I.$$

Lemma 3.6. *Let $L_{ab} = \lambda_{ab} y_a \neq 0$, and assume that the variable y_b does not appear in V . Then the conclusion of Theorem 3.1 holds.*

Proof. Assume that $L_{ab} = L_{12} = \lambda_{12} y_1$, and construct B^{12} . Since $d < m - 1$ there exists $y_w \neq y_2$ such that $y_w \notin B^{12}$. Let $Y_C = \{y_1, \dots, y_m\} \setminus \{y_2\} \setminus \{B^{12}\}$, and let C be the set of indexes of the variables in Y_C . Let $G_C = \{G_{2w} | w \in C\}$, where $G_{2w} = f_w = ((-1)^w \lambda_{12} - \mu_{2w}) y_w - \lambda_{1w} y_2$. By (5) we have that

$$(B^{12})(G_C) \subset I.$$

If for all $w \in C$ the coefficient of y_w in G_{2w} is not zero, then we are done.

Let $M_1 = \{w \in C | \mu_{2w} = (-1)^w \lambda_{12}\}$, $Y_{M_1} = \{y_w | w \in M_1\}$ and let $N_1 = N \setminus M_1$. We may assume that $M_1 \neq \emptyset$. If $w \in M_1$, then $L_{2w} = (-1)^w \lambda_{12} y_w \neq 0$ is a basis element of V . Since $d < m - 1$ we have that $N_1 \neq \emptyset$, and

$$(B^{12})(G_{N_1}) \subset I,$$

where the set $G_{N_1} = \{G_{2w} | w \in N_1\}$ consists of $|N_1|$ linearly independent linear forms. Up to possibly renaming the variables we may assume that if $w \in M_1$ and $v \in N_1$, then $v < w$.

Let Y_{C_1} be the set of monomials connected to Y_{M_1} in a finite number of steps, and let $C_1 \subset N_1$ be the set of indexes of the variables in Y_{C_1} . Then Y_{C_1} is part of a basis of V . Let $A_1 = N_1 \setminus C_1$. Since $d < m - 1$ we have that $A_1 \neq \emptyset$. By construction, for all $w \in M_1$ and for all $v \in A_1$ we have that $\lambda_{vw} = 0$.

By Remark 3.4 with $w \in M_1$ and $v \in A_1$, we obtain that $F_{2vw} = (-1)^{w+j-1} y_w G_{2v}$ if and only if the coefficient of y_w in G_{vw} is zero.

Let $M_2 = \{w \in M_1 | F_{2vw} = (-1)^{w+j-1} y_w G_{2v} \ \forall v \in A_1\}$, $Y_{M_2} = \{y_w | w \in M_2\}$ and let $N_2 = M_1 \setminus M_2$. If $N_2 = \emptyset$, then $(Y_{M_1})(G_{A_1}) \subset I$, where $G_{A_1} = \{G_{2v} | v \in A_1\}$. Then by Lemma 3.2 we have that

$$(B^{12}, Y_{M_1}, Y_{C_1})(G_{A_1}) \subset I,$$

and the conclusion follows.

So we may assume that $N_2 \neq \emptyset$. Let $w \in N_2$. We have that the coefficient of y_w in G_{vw} is not zero, for some $v \in A_1$. Let G_{N_2} be the set of such linear forms $\{G_{vw}\}$. By (6) we have that

$$(B^{12})(G_{N_1}, G_{N_2}) \subset I,$$

where the set $\{G_{N_1}, G_{N_2}\}$ consists of $|N_1| + |N_2|$ linearly independent linear forms. If $M_2 = \emptyset$, then we are done. Otherwise we repeat the procedure. Let Y_{C_2} be the set of monomials connected to Y_{M_2} in a finite number of steps, and let $C_2 \subset N_1 \cup N_2$ be the set of indexes of the variables in Y_{C_2} . Then $\{B^{12}, Y_{M_2}, Y_{C_2}\}$ is part of a basis of V . Let $A_2 = (N_1 \cup N_2) \setminus C_2$ and let $G_{A_2} \subset \{G_{N_1}, G_{N_2}\}$ be the set of corresponding linear forms. Since $d < m - 1$ we have that $A_2 \neq \emptyset$.

Applying Remark 3.4 with $w \in M_2$ and $u, v \in \{A_2\} \cup \{2\}$, we obtain that $F_{uvw} = (-1)^{w+j-1} y_w G_{uv}$ if and only if the coefficient of y_w in G_{uw} and G_{vw} is zero.

Let $M_3 = \{w \in M_2 | F_{uvw} = (-1)^{w+j-1} y_w G_{uv} \ \forall G_{uv} \in G_{A_2}\}$, and let $N_3 = M_2 \setminus M_3$. If $N_3 = \emptyset$, then $(Y_{M_2})(G_{A_2}) \subset I$. By Lemma 3.2 we have that

$$(B^{12}, Y_{M_2}, Y_{C_2})(G_{A_2}) \subset I,$$

and the conclusion follows.

So we may assume that $N_3 \neq \emptyset$. Let $w \in N_3$. We have that the coefficient of y_w in G_{uw} or in G_{vw} is not zero, for some u or v in A_2 . Let G_{N_3} be the set of such linear forms. We have that

$$(B^{12})(G_{N_1}, G_{N_2}, G_{N_3}) \subset I.$$

Repeating the argument we obtain that the conclusion of Theorem 3.1 is given by

$$(7) \quad (B^{12}, Y_{M_k}, Y_{C_k})(G_{A_k}) \subset I,$$

for some $k \geq 1$, or by

$$(8) \quad (B^{12})(\mathcal{G}) \subset I,$$

where \mathcal{G} consists of linear forms G_{uv} . □

Corollary 3.7. *Let $L_{ab} = \lambda_{ab} y_a \neq 0$, let $u \neq b$ and assume that the variable y_u does not appear in V . Then either the conclusion of Theorem 3.1 holds, or $(B^{ab})y_u \subset I$.*

Proof. Let $L_{ab} = L_{12} = \lambda_{12} y_1$. By Remark 3.5 we may assume that $y_2 \notin B^{12}$. Let $u \neq 2$. Then by (5) we have that $(B^{12})f_u \subset I$, where $f_u = (-1)^u \lambda_{12} y_u - (\lambda_{1u} + \lambda_{2u}) y_2$. If $\lambda_{1u} = \lambda_{2u} = 0$, then $(B^{12})y_u \subset I$. If $\lambda_{1u} \neq 0$, then $L_{1u} = \lambda_{1u} y_1 \neq 0$, and the conclusion follows from Lemma 3.6. If $\lambda_{2u} \neq 0$ we conclude similarly. □

Now we construct part of a basis of V consisting of monomials in the following way. If $L_{a_1 b_1} = \lambda_{a_1 b_1} y_{a_1} \neq 0$ we construct $B_1 = B^{a_1 b_1}$ and we assume that B_1 is *maximal*; that is, it is not contained in any bigger block starting with a monomial in \mathcal{L} . By Remark 3.5 we also assume that $y_{b_1} \notin B_1$. If among the linear forms in \mathcal{L} there is a nonzero monomial in one of the remaining variables, say $L_{a_2 b_2} = \lambda_{a_2 b_2} y_{a_2} \neq 0$ with $y_{a_2} \notin B_1$, we construct $B^{a_2 b_2}$ and we assume that it is maximal. We also assume that $y_{b_2} \notin B^{a_2 b_2}$. Let $B_2 = B^{a_2 b_2} \setminus (B^{a_2 b_2} \cap B_1)$. Proceeding in this way we construct $B^{a_1 b_1}, \dots, B^{a_k b_k}$ and B_1, \dots, B_k , where $\{B_1, \dots, B_k\}$ is part of a basis of V . We have that $y_{b_1} \notin \{B_2, \dots, B_k\}$, otherwise y_{a_1} would be connected to y_{b_1} and B_1 would not be maximal. More generally $y_{b_1}, \dots, y_{b_k} \notin \{B_1, \dots, B_k\}$.

Corollary 3.8. *If $\{B_1, \dots, B_k\}$, $k \geq 1$, is a basis of V , then the conclusion of Theorem 3.1 holds.*

Proof. By construction we have that $y_{b_1} \notin \{B_1, B_2, \dots, B_k\}$, and the conclusion follows from Lemma 3.6. \square

The following facts about blocks of monomials will be useful later.

Remark 3.9. Let $k \geq 2$, let $B_r, B_s \in \{B_1, \dots, B_k\}$ and assume that $r < s$; that is, B_s has been constructed after B_r . Let $L_{a_r a_s} = \lambda_{a_r a_s} y_{a_r} + \mu_{a_r a_s} y_{a_s}$. We have that $\mu_{a_r a_s} = 0$, otherwise $y_{a_s} \in B_r$. We also have that $\lambda_{a_r a_s} = 0$, otherwise $B^{a_r b_r}$ would not be maximal. Therefore $L_{a_r a_s} = 0$. Similarly $L_{a_r w} = 0$ for all $y_w \in B_s$. \square

Corollary 3.10. *In the notation of Remark 3.9, suppose that $y_{a_r} y_{a_s} \in I$. Then $(B_r)(B_s) \subset I$.*

Proof. By Lemma 3.2 we have that $(B_s)y_{a_r} \subset I$, since $L_{a_r w} = 0$ for all $y_w \in B_s$. Let $y_e \in B_r$. Then L_{ew} is a monomial in y_e for all $y_w \in B_s$, and so applying again Lemma 3.2 the conclusion follows. \square

Now we construct a basis $\{L_1, \dots, L_d\}$ of V consisting of $\{B_1, \dots, B_k\}$ as above and a set $L \subset \mathcal{L}$ of l binomials (necessarily in variables not involved in $\{B_1, \dots, B_k\}$). By Corollary 3.8 we may assume that $l \geq 1$. However, we may have that $k = 0$; that is, the basis is given by L .

We denote by V_L the variables involved in L , and by V_N the variables that do not appear in V . Let $|V_L| = s$. Then $|V_N| = m - d + l - s$.

Recall that if one among $y_{b_1}, \dots, y_{b_k} \in V_N$, then the conclusion of Theorem 3.1 follows from Lemma 3.6. So we may assume that $y_{b_1}, \dots, y_{b_k} \in V_L$, since by construction $y_{b_1}, \dots, y_{b_k} \notin \{B_1, \dots, B_k\}$. Then if $y_u \in V_N$, $y_u \neq y_{b_1}, \dots, y_{b_k}$, and so by Corollary 3.7 we may assume that $(B_1, \dots, B_k)(V_N) \subset I$.

Furthermore, if $L_{ab} \in L$ and $y_u \in V_N$, then by (1) we have $F_{abu} = \pm y_u L_{ab}$, so that $(L)(V_N) \subset I$. Therefore

$$(9) \quad (B_1, \dots, B_k, L)(V_N) \subset I.$$

If $l \geq s-1$, then the conclusion of Theorem 3.1 holds, since $d+(m-d+l-s) \geq m-1$. We may assume that $l \leq s-2$. In particular $l \geq 2$.

Next we need some facts about the binomials in L . If $C \subset \mathcal{L}$, we denote by V_C the set of variables involved in C .

Lemma 3.11. *Suppose that $\{L_1, \dots, L_u\} \subset \mathcal{L}$ are linearly independent binomials in q variables. Then $\{L_1, \dots, L_u\}$ is a disjoint union of subsets, $\{L_1, \dots, L_u\} = A \cup D_1 \cup \dots \cup D_p$, with $p = q - u$, such that $|V_{D_j}| = |D_j| + 1$ for all $j = 1, \dots, p$, and $|V_A| = |A|$.*

Proof. We start with $L_{ab} \in \{L_1, \dots, L_u\}$. At step 1 we add the binomials containing the variables y_a or y_b . At step $i \geq 2$ we add binomials containing variables introduced in the previous step. Since the binomials are linearly independent, the subset S thus constructed has the property that either $|V_S| = |S| + 1$, or $|V_S| = |S|$. Now if $\{L_1, \dots, L_u\} \setminus S \neq \emptyset$ we repeat the procedure.

Let D_j , $j = 1, \dots, p$, be the subsets with $|V_{D_j}| = |D_j| + 1$, and let A be the union of the remaining subsets. We have that $u = |A| + |D_1| + \dots + |D_p|$ and $q = |A| + (|D_1| + 1) + \dots + (|D_p| + 1)$, so that $q - u = p$. \square

By Lemma 3.11 we have that $L = D_0 \cup D_1 \cup \dots \cup D_p$, where $p = s - l \geq 2$, $|V_{D_j}| = |D_j| + 1$ for all $j = 1, \dots, p$, and $|V_{D_0}| = |D_0|$. We may have $D_0 = \emptyset$.

Lemma 3.12. *Let $L_{ef} \in \mathcal{L}$. In the above set-up, we have that $L_{ef} = 0$ if $y_e \in V_{D_j}$ for some $j = 1, \dots, p$, and $y_f \in V_L \setminus V_{D_j}$.*

Proof. For simplicity of notation, let $L = \{L_1, \dots, L_l\}$ and $D_j = \{L_1, \dots, L_r\}$. Write $L_{ef} = \alpha_1 L_1 + \dots + \alpha_r L_r + \alpha_{r+1} L_{r+1} + \dots + \alpha_l L_l$. It suffices to show that $\alpha_1 = \dots = \alpha_r = 0$. By construction $L_1 = L_{i_1 i_2}$ is a binomial in two variables y_{i_1} and y_{i_2} , and for $2 \leq m \leq r$, $L_m = L_{a_m i_{m+1}}$ is a binomial in y_{a_m} and $y_{i_{m+1}}$, where $a_m \in \{i_1, \dots, i_m\}$. Then $e = i_c$ for some $1 \leq c \leq r + 1$. It follows that

$$(10) \quad L_{ef} = \alpha_1 L_{i_1 i_2} + \alpha_2 L_{a_2 i_3} + \dots + \alpha_{c-1} L_{a_{c-1} i_c} + \alpha_{r+1} L_{r+1} + \dots + \alpha_l L_l,$$

and that $L_{ef} = 0$, if $c = 1, 2$. Let $c \geq 3$. There exists $i_k \in \{i_1, \dots, i_{c-1}\}$ such that $i_k \neq a_2, \dots, a_{c-1}$. Therefore y_{i_k} appears in (10) only once; in $L_{a_{k-1} i_k}$ if $k \geq 3$, or in $L_{i_1 i_2}$ if $k = 1, 2$. So if $k \geq 2$ we have that $\alpha_{k-1} = 0$, and if $k = 1$ we have that $\alpha_1 = 0$. Now we repeat the procedure to obtain that $\alpha_1 = \dots = \alpha_r = 0$. \square

It follows from Lemma 3.12 that $L_{ac} = 0$ if $y_a \in V_{D_j}$ for some $j = 0, \dots, p$, and $y_c \in V_L \setminus V_{D_j}$. Hence if $y_a, y_b \in V_{D_j}$ and $y_c \in V_L \setminus V_{D_j}$, by (1) we have that $F_{abc} = \pm y_c L_{ab}$. Therefore for all $j = 0, \dots, p$,

$$(11) \quad (D_j)(V_L \setminus V_{D_j}) \subset I.$$

Remark 3.13. Suppose that the basis of V consists only of binomials. We have that

$$(12) \quad (D_1)(V_L \setminus V_{D_1}, V_N) \subset I.$$

Let $|D_1| = r$. The conclusion of Theorem 3.1 holds, since $r + (s - r - 1) + (m - s) = m - 1$.

Lemma 3.14. *Let $\{B_1, \dots, B_k, L\}$ be a basis of V and suppose that $L = U \cup Z$ where $L_{uz} = 0$ if $y_u \in V_U$ and $y_z \in V_Z$. Let $B \in \{B_1, \dots, B_k\}$ be obtained from $L_{ab} = \lambda_{ab}y_a \neq 0$ and assume that $y_b \in V_U$. Suppose that there exists $y_t \in V_Z$ with $\lambda_{at} \neq 0$. Then the conclusion of Theorem 3.1 holds.*

Proof. Assume that $L_{ab} = L_{12} = \lambda_{12}y_1 \neq 0$. Then $y_2 \in V_U$ and there exists $y_t \in V_Z$ such that $\lambda_{1t} \neq 0$.

Let s be such that $y_s \in V_N \cup V_Z$. Then by (2) $f_s = (-1)^s \lambda_{12}y_s - \lambda_{1s}y_2$, since $L_{2s} = 0$. Let $f_{N \cup Z}$ be the set of such linear forms.

Let s be such that $y_s \in V_U$ and $y_s \neq y_2$. Then by (3) $G_{st} = (-1)^{s-1} \lambda_{1t}y_s + (-1)^t \lambda_{1s}y_t$, since $L_{st} = 0$. Let G_U be the set of such linear forms.

Let $V_C = \{B_1, \dots, B_k\} \setminus \{B^{ab}\}$, and let C be the set of indexes of the variables in V_C . Let $f_C = \{f_s | s \in C\}$, where $f_s = ((-1)^s \lambda_{12} - \mu_{2s})y_s - \lambda_{1s}y_2$, since $\lambda_{2s} = 0$. By (6) we have that

$$(B^{12})(f_C, f_{N \cup Z}, G_U) \subset I,$$

where $\{f_{N \cup Z}, G_U\}$ are linearly independent.

If for all $s \in C$ the coefficient of y_s in f_s is not zero, then we are done.

Otherwise let $M_1 = \{s \in C | \mu_{2s} = (-1)^s \lambda_{12}\}$ and we proceed as in the proof of Lemma 3.6. \square

Corollary 3.15. *Let $B \in \{B_1, \dots, B_k\}$ be obtained from $L_{ab} = \lambda_{ab}y_a \neq 0$ and assume that $y_b \in V_{D_j}$ for some $j = 0, \dots, p$. Then either the conclusion of Theorem 3.1 holds, or $(B^{ab})(V_L \setminus V_{D_j}) \subset I$.*

Proof. Let $y_c \in V_L \setminus V_{D_j}$. Then by (1) $F_{abc} = \pm y_a L_{bc} \pm y_b L_{ac} \pm y_c L_{ab} = \pm \lambda_{ac} y_a y_b \pm \lambda_{ab} y_a y_c$. If $\lambda_{ac} \neq 0$ the conclusion follows by Lemma 3.14. If $\lambda_{ac} = 0$, then $y_a y_c \in I$, and so by Lemma 3.2 $(B^{ab})y_c \subset I$. \square

Back to the proof of Theorem 3.1, recall that a basis of V is given by

$$\{B_1, \dots, B_k, D_0, D_1, \dots, D_p\},$$

with $p \geq 2$ and $k \geq 1$, by Remark 3.13. We have $L = D_0 \cup D_1 \cup \dots \cup D_p$, $|L| = l$, $|V_L| = s$. Let $|V_{D_j}| = s_j$, for $0 \leq j \leq p$. Then $|D_0| = s_0$ and $|D_j| = s_j - 1$ for $1 \leq j \leq p$. Let $q \in \{1, \dots, k\}$. If B_q is obtained from $L_{a_q b_q} = \lambda_{a_q b_q} y_{a_q} \neq 0$, we have that $y_{b_q} \in V_{D_{j_q}}$ for some $j_q \in \{0, \dots, p\}$.

By (9), (11) and Corollary 3.15 we may assume that

$$(13) \quad (B_1, D_{j_1})(V_N, V_L \setminus V_{D_{j_1}}) \subset I.$$

If $k = 1$ the conclusion of Theorem 3.1 holds, so we may assume that $k \geq 2$.

We divide $\{B_2, \dots, B_k\}$ in two groups. Let $\{B_{d_2}, \dots, B_{d_r}\}$ be such that there exist $y_{t_2}, \dots, y_{t_r} \in V_{D_{j_1}}$ such that $L_{a_{d_2} t_2}, \dots, L_{a_{d_r} t_r}$ are nonzero monomials in $y_{a_{d_2}}, \dots, y_{a_{d_r}}$,

respectively. Let $\{B_{d_{r+1}}, \dots, B_{d_k}\}$ be the remaining blocks. Then for every $y_t \in V_{D_{j_1}}$ we have that $L_{a_{d_{r+1}}t} = \dots = L_{a_{d_k}t} = 0$. We may assume that

$$(14) \quad (B_1, B_{d_2}, \dots, B_{d_r}, D_{j_1})(V_N, V_L \setminus V_{D_{j_1}}) \subset I.$$

Corollary 3.16. *In the above notation, assume that $\bar{B}_1 \in \{B_1, B_{d_2}, \dots, B_{d_r}\}$ and $\bar{B}_2 \in \{B_{d_{r+1}}, \dots, B_{d_k}\}$. Then $(\bar{B}_1)(\bar{B}_2) \subset I$.*

Proof. Assume that \bar{B}_1 is obtained from $L_{cd} = \lambda_{cd}y_c$ with $y_d \in V_{D_{j_1}}$, and \bar{B}_2 is obtained from $L_{ef} = \lambda_{ef}y_e$. By construction we have that $L_{de} = 0$, and by Remark 3.9 we have that $L_{ce} = 0$. Then by (1), $F_{cde} = \pm \lambda_{cd}y_cy_e$. Therefore $y_cy_e \in I$, and the conclusion follows from Corollary 3.10. \square

Corollary 3.17. *In the above notation, assume that $\bar{B}_2 \in \{B_{d_{r+1}}, \dots, B_{d_k}\}$. Then $(\bar{B}_2)(D_{j_1}) \subset I$.*

Proof. Assume that \bar{B}_2 is obtained from $L_{ef} = \lambda_{ef}y_e$. Let $L_{ab} \in D_{j_1}$. Then $L_{ae} = L_{be} = 0$, and so by (1) $F_{abe} = \pm y_e L_{ab}$. Since $y_e(D_{j_1}) \subset I$, by Lemma 3.2 we have that $(\bar{B}_2)(D_{j_1}) \subset I$. \square

Finally, by (14), Corollary 3.16 and Corollary 3.17, we have that

$$(15) \quad (B_1, B_{d_2}, \dots, B_{d_r}, D_{j_1})(V_N, V_L \setminus V_{D_{j_1}}, B_{d_{r+1}}, \dots, B_{d_k}) \subset I.$$

Now, if $j_1 \neq 0$ the conclusion of Theorem 3.1 follows since $(d-l) + (s_{j_1} - 1) + (m-d+l-s) + (s-s_{j_1}) = m-1$. Similarly, if $j_1 = 0$ the conclusion follows since $(d-l) + s_0 + (m-d+l-s) + s - s_0 = m$. \square

4. MAIN RESULT

In this section we start the proof of the main theorem, stated in Section 1 as Theorem 1.3.

Theorem 4.1. *Let X be a set of distinct points spanning \mathbb{P}^n . Fix $i = 0, \dots, n-2$. Assume that:*

- (1) *There exist $n-i+1$ points of X on a \mathbb{P}^{n-i-1} ,*
- (2) *$n-i$ points of X are never on a \mathbb{P}^{n-i-2} ,*
- (3) *$a_{n-1} \neq 0$.*

Then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k+r = n$.

Proof. Since $n-i+1$ points are in \mathbb{P}^{n-i-1} they must span it, otherwise we get $n-i+1$ points on \mathbb{P}^{n-i-2} . After a change of coordinates we may assume that the coordinate points are on X and that the linear space $x_{n-i} = x_{n-i+1} = \dots = x_n = 0$ contains $n-i+1$ points of X . This linear space contains $n-i$ coordinate points, so it contains an “extra point” $Q = (q_0, \dots, q_{n-i-1}, 0, \dots, 0)$. Notice that $q_l \neq 0$ for all $l = 0, \dots, n-i-1$, otherwise we would have $n-i$ points in \mathbb{P}^{n-i-2} .

Let $0 \leq a < b \leq n - i - 1$ and $n - i \leq j \leq n$. Consider the quadrics

$$F_{abj} = \lambda_{abj}x_ax_b + \mu_{abj}x_ax_j + \nu_{abj}x_bx_j,$$

defined in Section 2. Since $F_{abj}(Q) = 0$, we have that

$$F_{abj} = x_j L_{ab}^j,$$

where L_{ab}^j is a linear form in x_a and x_b .

Using Remark 2.2 we obtain the following result, which is a generalization of Claim 5 of [2].

Lemma 4.2.

(1) *Let $0 \leq a < b < c \leq n - i - 1$ and $n - i \leq j \leq n$. Then*

$$F_{abc} = (-1)^{a+j}x_a L_{bc}^j + (-1)^{b+j-1}x_b L_{ac}^j + (-1)^{c+j}x_c L_{ab}^j.$$

(2) *Fix d and e such that $n - i \leq d < e \leq n$. Then there exists $P_{de} = \lambda_{de}x_d + \mu_{de}x_e$ such that for all $s = 0, \dots, n - i - 1$,*

$$F_{sde} = (-1)^s x_s P_{de} + \nu_{sde} x_d x_e.$$

(3) *Let $0 \leq a < b \leq n - i - 1$ and $n - i \leq d < e \leq n$. We have that*

$$(-1)^a x_a \nu_{bde} + (-1)^{b-1} x_b \nu_{ade} + (-1)^{d-2} L_{ab}^e + (-1)^{e-3} L_{ab}^d = 0.$$

Let $j \in \{n - i, \dots, n\}$ and let W_j be the k -vector space generated by the linear forms L_{ab}^j .

The proof of Theorem 4.1 consists of two main steps:

- (1) If $W_j = 0$ for all $j = n - i, \dots, n$, then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k + r = n$.
- (2) If $W_j \neq 0$ for some $j \in \{n - i, \dots, n\}$, then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k + r = n$.

4.1. Proof of Step 1. Suppose that $W_j = 0$ for all $j = n - i, \dots, n$. Then $F_{abc} = 0$, if $0 \leq a < b \leq n - i - 1$ and $n - i \leq c \leq n$. If $0 \leq a < b < c \leq n - i - 1$, we also have that $F_{abc} = 0$, by Lemma 4.2 (1). In particular, if $i = 0$ we have that $\alpha = 0$, a contradiction to Remark 2.1, since $a_{n-1} \neq 0$. So we may assume that $i \geq 1$.

If $0 \leq a \leq n - i - 1$ and $n - i \leq b < c \leq n$, by Lemma 4.2 (2) and (3) we have that

$$F_{abc} = (-1)^a x_a P_{bc}.$$

Therefore, if $P_{bc} = 0$ for all $n - i \leq b < c \leq n$, we have $F_{abc} = 0$ for all $0 \leq a \leq n - i - 1$ and all $n - i \leq b < c \leq n$. It follows from Remark 2.2 that $F_{abc} = 0$ for all $n - i \leq a < b < c \leq n$. Hence $\alpha = 0$, a contradiction.

Let W be the k -vector space generated by the linear forms P_{bc} with $n - i \leq b < c \leq n$, and let $d := \dim W > 0$.

We have that

$$(x_0, \dots, x_{n-i-1})W \subset I.$$

We may assume that $d < i$, otherwise $X \subset \mathbb{P}^i \cup \mathbb{P}^{n-i}$ and we conclude. Then by Theorem 3.1 there exist t , with $0 < t \leq d$, linear forms L_1, \dots, L_t , which are part of a basis of W , and linearly independent linear forms h_1, \dots, h_{i-t} such that

$$(L_1, \dots, L_t)(h_1, \dots, h_{i-t}, x_0, \dots, x_{n-i-1}) \subset I.$$

Hence $X \subset \mathbb{P}^{n-t} \cup \mathbb{P}^t$ and the conclusion of Theorem 4.1 holds. This concludes the proof of Step 1.

5. PROOF OF THE MAIN THEOREM

In this section we complete the proof of Theorem 4.1 by proving Step 2.

Fix $j \in \{n-i, \dots, n\}$ such that $W_j \neq 0$. Recall that W_j is the k -vector space generated by the linear forms L_{ab}^j , where $F_{abj} = x_j L_{ab}^j$, for $0 \leq a < b \leq n-i-1$.

Let \mathcal{L}^j denote the set of linear forms $\{L_{ab}^j \mid 0 \leq a < b \leq n-i-1\}$.

If $0 \leq d < e < f \leq n$, we have

$$F_{def} = \lambda_{def} x_d x_e + \mu_{def} x_d x_f + \nu_{def} x_e x_f.$$

Remark 5.1. Let $0 \leq d < e < f < g \leq n$. By Remark 2.2 we have the following equations:

$$(16) \quad (-1)^d \lambda_{efg} + (-1)^{e-1} \lambda_{dfg} + (-1)^{f-2} \lambda_{deg} = 0.$$

$$(17) \quad (-1)^d \mu_{efg} + (-1)^{e-1} \mu_{dfg} + (-1)^{g-3} \lambda_{def} = 0.$$

$$(18) \quad (-1)^d \nu_{efg} + (-1)^{f-2} \mu_{deg} + (-1)^{g-3} \mu_{def} = 0.$$

$$(19) \quad (-1)^{e-1} \nu_{dfg} + (-1)^{f-2} \nu_{deg} + (-1)^{g-3} \nu_{def} = 0.$$

□

In what follows we give an explicit description of quadrics that are multiple of x_j .

Let $\{d, e \mid d \neq e\} \subset \{0, \dots, n\}$, and assume that $F_{dej} \in (x_j)$. Without loss of generality, assume that $d < e < j$. Then $\lambda_{dej} = 0$, and

$$F_{dej} = x_j(\mu_{dej} x_d + \nu_{dej} x_e) = x_j H_{de},$$

where

$$H_{de} = \mu_{dej} x_d + \nu_{dej} x_e.$$

In particular, if $0 \leq d < e \leq n-i-1$, $H_{de} = L_{de}^j$.

Lemma 5.2. Let $\{d, e, f, g\} \subset \{0, \dots, n\}$. Let T_{de} be a linear form in x_d and x_e . Assume that $x_g T_{de} \in I$, $x_f T_{de} \notin I$, and that $F_{deg} \in (x_g)$. Then $F_{dfg} \in (x_g)$ and $F_{efg} \in (x_g)$.

Proof. There exists a point E such that $x_f(E) \neq 0$, $T_{de}(E) \neq 0$, and $x_g(E) = 0$. We may assume that $d < e < f < g$. We have $\lambda_{deg} = 0$. If $x_d(E) \neq 0$, $F_{dfg}(E) = 0$ implies that $\lambda_{dfg} = 0$. Similarly if $x_e(E) \neq 0$, then $\lambda_{efg} = 0$. Now (16) implies that $\lambda_{dfg} = \lambda_{efg} = 0$. \square

Corollary 5.3. *Let $\{d, e, f\} \subset \{0, \dots, n\}$. Assume that $F_{dej} = x_j H_{de}$ and $x_f H_{de} \notin I$. Then $F_{dfj} = x_j H_{df}$ and $F_{efj} = x_j H_{ef}$. Furthermore, if $F_{def} \in (x_f)$, we have that the coefficient of x_f in H_{df} or in H_{ef} is not zero.*

Proof. The first assertion follows from Lemma 5.2. We may assume that $d < e < f < j$. We have that $F_{dfj} = x_j(\mu_{dfj}x_d + \nu_{dfj}x_f)$, $F_{efj} = x_j(\mu_{efj}x_e + \nu_{efj}x_f)$ and $\lambda_{def} = 0$. If $(\nu_{dfj}, \nu_{efj}) = (0, 0)$, by (18) and (19) we have that $\mu_{def} = (-1)^{f-j}\mu_{dej}$, and $\nu_{def} = (-1)^{f-j}\nu_{dej}$. Then $F_{def} = (-1)^{f-j}x_f H_{de}$, a contradiction, since $x_f H_{de} \notin I$. \square

Theorem 5.4. *If $\dim W_j \geq n - i - 1$, then $X \subset \mathbb{P}^k \cup \mathbb{P}^r$ for some positive integers k and r such that $k + r = n$.*

Proof. We have that $x_j W_j \subset I$. If $(x_{n-i}, \dots, x_n)W_j \subset I$, the conclusion follows. Let $A_0 = \{x_{n-i}, \dots, x_n\}$, and let $V_1 = \{x_u \in A_0 \mid x_u L_{ab}^j \notin I \text{ for some } L_{ab}^j \in W_j\}$. We may assume that $V_1 \neq \emptyset$. Since $F_{abu} = x_u L_{ab}^u$, by Corollary 5.3 we have that $x_u \in V_1$ yields a linear form $H_{c_1 u}$, $c_1 \in \{a, b\}$, with coefficient of x_u different from zero, such that $F_{c_1 u j} = x_j H_{c_1 u}$. Let H_{V_1} be the set of such linear forms $H_{c_1 u}$. We have that $x_j(W_j, H_{V_1}) \subset I$. Let $A_1 = A_0 \setminus V_1$. If $A_1 = \{x_j\}$, then $X \subset \mathbb{P}^{n-1} \cup \mathbb{P}^1$, so we may assume that $\{x_j\} \subsetneq A_1$. If $(A_1)(H_{V_1}) \subset I$, the conclusion follows.

Let $V_2 = \{x_v \in A_1 \mid x_v H_{c_1 u} \notin I \text{ for some } H_{c_1 u} \in H_{V_1}\}$. Since $x_u L_{ab}^j \notin I$, and $x_v L_{ab}^j \in I$, by Lemma 5.2 we have that $F_{c_1 u v} \in (x_v)$. By Corollary 5.3, $x_v \in V_2$ yields a linear form $H_{c_2 v}$, $c_2 \in \{c_1, u\}$, with coefficient of x_v different from zero, such that $F_{c_2 v j} = x_j H_{c_2 v}$. Let H_{V_2} be the set of such linear forms.

Let $l \geq 2$, and $A_l = A_{l-1} \setminus V_l$. We may assume that

$$x_j(W_j, H_{V_1}, \dots, H_{V_l}) \subset I,$$

and

$$(A_l)(W_j, H_{V_1}, \dots, H_{V_{l-1}}) \subset I.$$

If $(A_l)(H_{V_l}) \subset I$, the conclusion follows.

Let $V_{l+1} = \{x_z \in A_l \mid x_z H_{c_l w} \notin I \text{ for some } H_{c_l w} \in H_{V_l}\}$. By inductively applying Lemma 5.2 we have that $F_{c_l w z} \in (x_z)$. By Corollary 5.3, x_z yields a linear form $H_{c_{l+1} z}$, $c_{l+1} \in \{c_l, w\}$, with coefficient of x_z different from zero, such that $F_{c_{l+1} z j} = x_j H_{c_{l+1} z}$. Repeating the procedure we obtain the desired conclusion. \square

By Theorem 5.4 we may assume that $\dim W_j < n - i - 1$. Then by Theorem 3.1 and its proof (with $\{y_1, \dots, y_m\} = \{x_0, \dots, x_{n-i-1}\}$), there exist t , with $0 < t \leq \dim W_j$, linear forms L_1, \dots, L_t , which are part of a basis of W_j , and linearly independent linear

forms $h_1, \dots, h_{n-i-1-t}$, in variables x_0, \dots, x_{n-i-1} , but not involved in $\{L_1, \dots, L_t\}$, such that

$$(20) \quad (L_1, \dots, L_t)(h_1, \dots, h_{n-i-1-t}) \subset I.$$

By construction we may assume that $\{L_1, \dots, L_t\} \subset \mathcal{L}^j$. If $1 \leq p \leq n-i-1-t$, then h_p “contributes” the variable x_{l_p} ; that is, the coefficient of x_{l_p} in h_p is not zero, and $l_p \neq l_q$ if $p \neq q$.

Recall that $x_j(L_1, \dots, L_t) \subset I$. If

$$(L_1, \dots, L_t)(h_1, \dots, h_{n-i-1-t}, x_{n-i}, \dots, x_n) \subset I,$$

then the conclusion of Theorem 4.1 holds.

As in the proof of Theorem 5.4, let $A_0 = \{x_{n-i}, \dots, x_n\}$, and let

$$V_1 = \{x_u \in A_0 \mid x_u L_{ab}^j \notin I \text{ for some } L_{ab}^j \in \{L_1, \dots, L_t\}\}.$$

We may assume that $V_1 \neq \emptyset$. Since $F_{abu} = x_u L_{ab}^u$, by Corollary 5.3 we have that $x_u \in V_1$ yields a linear form $H_{c_1 u}$, $c_1 \in \{a, b\}$, with coefficient of x_u different from zero, such that $F_{c_1 u j} = x_j H_{c_1 u}$. Let H_{V_1} be the set of such linear forms $H_{c_1 u}$.

Fix $u \in V_1$. By Lemma 4.2 (2) we have that for all $s = 0, \dots, n-i-1$,

$$F_{suj} = x_j((-1)^{c_1+s} \mu_{c_1 u j} x_s + \nu_{suj} x_u).$$

If $\mu_{c_1 u j} \neq 0$, then $H_{l_1 u}, \dots, H_{l_{n-i-1-t} u}$ are linearly independent, and

$$x_j(L_1, \dots, L_t, H_{V_1}, H_{l_1 u}, \dots, H_{l_{n-i-1-t} u}) \subset I.$$

Let $H_{V_1'} = \{H_{V_1}, H_{l_1 u}, \dots, H_{l_{n-i-1-t} u}\}$, and let $A_1 = A_0 \setminus V_1$. If $(A_1)(H_{V_1'}) \subset I$, the conclusion of Theorem 4.1 follows.

Notice that if $\{u, v\} \subset \{n-i, \dots, n\}$, then by Lemma 4.2 (2) we have $F_{c_1 uv} \in (x_v)$ if and only if $F_{l_p uv} \in (x_v)$ for some $1 \leq p \leq n-i-1-t$. Now we proceed as in the proof of Theorem 5.4, and the conclusion of Theorem 4.1 holds.

We may assume that $\mu_{c_1 u j} = 0$ for all $x_u \in V_1$. Then $F_{c_1 u j} = \nu_{c_1 u j} x_j x_u$, so that $x_j x_u \in I$. If

$$(A_1, h_1, \dots, h_{n-i-1-t})(L_1, \dots, L_t, V_1) \subset I,$$

then the conclusion of Theorem 4.1 follows.

Section 5.1 below shows that $(V_1)(h_1, \dots, h_{n-i-1-t}) \subset I$. Therefore we may assume that $x_u x_v \notin I$ for some $x_u \in V_1$ and some $x_v \in A_1$; that is, the set $V_2 = \{x_v \in A_1 \mid x_v H_{c_1 u} \notin I \text{ for some } H_{c_1 u} \in H_{V_1}\}$ is not empty. As in the proof of Theorem 5.4, we have that $x_v \in V_2$ yields a linear form $H_{c_2 v}$, $c_2 \in \{c_1, u\}$, with coefficient of x_v different from zero, such that $F_{c_2 v j} = x_j H_{c_2 v}$. Let H_{V_2} be the set of such linear forms.

Fix $v \in V_2$. If $c_2 = c_1 \in \{a, b\}$, Lemma 4.2 (2) implies that for all $s = 0, \dots, n-i-1$, $F_{svj} = x_j(\pm \mu_{c_2 v j} x_s + \nu_{svj} x_v)$. If $\mu_{c_2 v j} \neq 0$, then $H_{l_1 v}, \dots, H_{l_{n-i-1-t} v}$ are linearly independent, and

$$x_j(L_1, \dots, L_t, H_{V_1}, H_{V_2}, H_{l_1 v}, \dots, H_{l_{n-i-1-t} v}) \subset I.$$

Let $H_{V'_2} = \{H_{V_2}, H_{l_1v}, \dots, H_{l_{n-i-1-t}v}\}$, and let $A_2 = A_1 \setminus V_2$. If $(A_2)(H_{V'_2}) \subset I$, the conclusion of Theorem 4.1 follows. Otherwise we proceed as in the proof of Theorem 5.4.

Therefore if $c_2 = c_1$ we may assume that $\mu_{c_2vj} = 0$. Then $F_{c_2vj} = \nu_{c_2vj}x_jx_v$, so that $x_jx_v \in I$. If $c_2 = u$, then $F_{uvj} = x_j(\mu_{uvj}x_u + \nu_{uvj}x_v)$, with $\nu_{uvj} \neq 0$. Since $x_jx_u \in I$, we have that $x_jx_v \in I$. Furthermore, $F_{c_1vj} = x_j(\mu_{c_1vj}x_{c_1})$. As above, if $\mu_{c_1vj} \neq 0$, then $H_{l_1v}, \dots, H_{l_{n-i-1-t}v}$ are linearly independent, and we proceed as in the proof of Theorem 5.4.

Hence we may assume that the condition “ $x_u x_v \notin I$ for some $x_u \in V_1$ and some $x_v \in A_1$ ” yields $x_jx_v \in I$. We say that x_v is *introduced from* x_u .

Section 5.1 shows that $(V_2)(h_1, \dots, h_{n-i-1-t}) \subset I$. If

$$(A_2, h_1, \dots, h_{n-i-1-t})(L_1, \dots, L_t, V_1, V_2) \subset I,$$

the conclusion of Theorem 4.1 follows. Otherwise we repeat the argument. Proceeding in this way, at step $l \geq 1$ either we conclude as in the proof of Theorem 5.4, or we introduce a new set of monomials V_l such that $x_j(V_l) \subset I$ and $(V_l)(h_1, \dots, h_{n-i-1-t}) \subset I$. Furthermore, by inductively applying (16), we have that if $x_z \in V_l$, then $F_{c_1zj} \in (x_j)$. Therefore we assume that $\mu_{c_1zj} = 0$; that is, in the notation of Lemma 4.2 (2), $P_{zj} = 0$. This procedure has to terminate in a finite number of steps, and so the conclusion of Theorem 4.1 holds.

5.1. To conclude the proof of Theorem 4.1 we need to show that for each $l \geq 1$, the set V_l has the property that $(V_l)(h_1, \dots, h_{n-i-1-t}) \subset I$.

We follow the proof of Theorem 3.1 and we consider the explicit description of $h_1, \dots, h_{n-i-1-t}$ in (20). Recall that $\{y_1, \dots, y_m\} = \{x_0, \dots, x_{n-i-1}\}$.

If $0 \leq a < b \leq n-i-1$ and $n-i \leq u \leq n$, we have that $F_{abu} = x_u L_{ab}^u$. Let

$$L_{ab}^u = \lambda_{ab}^u x_a + \mu_{ab}^u x_b.$$

First we summarize some general facts that will be used often.

Remark 5.5. Let $0 \leq a < b \leq n-i-1$, and $n-i \leq d < e \leq n$. By Lemma 4.2 (3) we have that $\lambda_{ab}^d = (-1)^{a-e} \nu_{bde} + (-1)^{d-e} \lambda_{ab}^e$ and $\mu_{ab}^d = (-1)^{b-1-e} \nu_{ade} + (-1)^{d-e} \mu_{ab}^e$.

Lemma 5.6. Let $\{c, f, g\} \subset \{0, \dots, n-i-1\}$. Let T_{fg} be a linear form in x_f and x_g . Suppose that $x_c T_{fg} \in I$, and that L_{cf}^j, L_{cg}^j are monomials in x_c . Let $u \in \{n-i, \dots, n\} \setminus \{j\}$ and suppose that the coefficient of $x_u x_j$ in $F_{cu j}$ is not zero. Then $x_u T_{fg} \in I$.

Proof. Assume $c < f < g < u < j$, so that $\nu_{cu j} \neq 0$. By Remark 5.5 we have that $\mu_{cf}^u = \pm \nu_{cu j}$, $\mu_{cg}^u = \pm \nu_{cu j}$. If $x_u T_{fg} \notin I$, there exists a point E such that $x_u(E) \neq 0$, $T_{fg}(E) \neq 0$, and $x_c(E) = 0$. Without loss of generality assume that $x_f(E) \neq 0$. Recall that $F_{cfu} = x_u(\lambda_{cf}^u x_c + \mu_{cf}^u x_f)$. Then $F_{cfu}(E) = 0$ implies that $\mu_{cf}^u = 0$, a contradiction. \square

Lemma 5.7. *Let $u \in \{n-i, \dots, n\}$ and assume that $x_u L_{ab}^j \notin I$ for some $L_{ab}^j \in \{L_1, \dots, L_t\}$. Let T_{fg} be a linear form in x_f and x_g , where $\{f, g\} \subset \{0, \dots, n-i-1\}$. Assume that $x_a T_{fg} \in I$ and $x_b T_{fg} \in I$, that L_{af}^j, L_{ag}^j are monomials in x_a , and L_{bf}^j, L_{bg}^j are monomials in x_b . Suppose $x_z \in V_l$ is introduced inductively from $x_u \in V_1$. Then $x_z T_{fg} \in I$.*

Proof. We proceed by induction on l . If $l = 1$, then $z = u$. Since $x_j L_{ab}^j \in I$ and $x_u L_{ab}^j \notin I$, by Corollary 5.3 we have that $(\nu_{auj}, \nu_{buj}) \neq (0, 0)$. Then $x_u T_{fg} \in I$ by Lemma 5.6. Now suppose that $l > 1$ and that $x_z \in V_l$ is introduced because $x_w x_z \notin I$ for some $x_w \in V_{l-1}$; that is, $x_z H_{c_{l-1}w} \notin I$ for some $H_{c_{l-1}w} \in H_{V_{l-1}}$. Assuming $c_{l-1} < w < z < j$, by construction we have that $(\nu_{c_{l-1}zj}, \nu_{wzj}) \neq (0, 0)$. If $c_{l-1} = c_1 \in \{a, b\}$ and $\nu_{c_1zj} \neq 0$, we conclude by Lemma 5.6. Otherwise $\nu_{c_1zj} \neq 0$, where $x_{c_1} \in V_p$ for some $p < l$, so that $x_{c_1} T_{fg} \in I$. Assume by contradiction that $x_z T_{fg} \notin I$. By Lemma 5.2 and Lemma 4.2 (2) we have that $\mu_{fc_1z} = \mu_{gc_1z} = \mu_{c_1c_1z} = 0$. Recall that we are assuming $\mu_{c_1c_1j} = 0$. Then by (18) we have that $\nu_{c_1zj} = 0$, a contradiction. \square

Now suppose that (20) is given by (4) of Remark 3.5,

$$(B^{12})(Y_C) \subset I.$$

Let $x_u \in V_1$ be such that $x_u L_{ab}^j \notin I$. By construction we have that $x_a, x_b \in B^{12}$. If $s \in C$ let $T_s = x_s$. Then by Lemma 5.7, we have that $(V_l)(Y_C) \subset I$, as desired.

Next we suppose that (20) is given by (7) of Lemma 3.6,

$$(B^{12}, Y_{M_k}, Y_{C_k})(G_{A_k}) \subset I,$$

for some $k \geq 1$. For simplicity of notation we may assume that $x_1 = y_1$ and $x_2 = y_2$. Recall that G_{A_k} consists of forms G_{fg} defined in (3),

$$G_{fg} = ((-1)^{f-1} \lambda_{1g}^j - \lambda_{fg}^j) x_f + ((-1)^g \lambda_{1f}^j - \mu_{fg}^j) x_g.$$

Therefore (7) includes equations of type (8).

Let $x_u \in V_1$. By Lemma 5.7, it suffices to consider the cases $x_u L_{12}^j \notin I$, where $L_{12}^j = \lambda_{12}^j x_1$, and $x_u L_{2q}^j \notin I$, with $x_q \in Y_{M_k}$; that is, $L_{2q}^j = (-1)^q \lambda_{12}^j x_q$.

Suppose $x_u L_{12}^j \notin I$. Let $z \in V_l$, $l \geq 1$, be obtained inductively from x_u . We have that $F_{fgz} = x_z(\lambda_{fg}^z x_f + \mu_{fg}^z x_g)$. By Remark 5.5, assuming that $f < g < z < j$, we have that

$$\lambda_{fg}^z = (-1)^{f-j} \nu_{gzj} + (-1)^{z-j} \lambda_{fg}^j$$

and

$$\mu_{fg}^z = (-1)^{g-1-j} \nu_{fzj} + (-1)^{z-j} \mu_{fg}^j.$$

Since $x_1 G_{fg} \in I$, if $\nu_{1zj} \neq 0$, we have that $x_z G_{fg} \in I$ by Lemma 5.6. Therefore we may assume that $\nu_{1zj} = 0$.

First we show that $x_u G_{fg} \in I$. Since $\nu_{1uj} = 0$, by Remark 5.5 we have that $\mu_{1f}^u = \mu_{1g}^u = 0$. Then $F_{1fu} = F_{1gu} = 0$, since $x_1 x_u \notin I$. Then by Remark 5.5 we have that $\nu_{fuj} = (-1)^u \lambda_{1f}^j$ and $\nu_{guj} = (-1)^u \lambda_{1g}^j$. It follows that

$$F_{fgu} = (-1)^{u-j+1} x_u G_{fg},$$

and so $x_u G_{fg} \in I$.

Now suppose $x_v \in V_2$ is introduced because $x_u x_v \notin I$. If $\mu_{1uv} \neq 0$, we have that $x_v G_{fg} \in I$ by Lemma 5.2 and Lemma 4.2 (2). Therefore we may assume that $\mu_{1uv} = 0$. Since $x_1 x_v \in I$ and $x_1 x_u \notin I$, we have that $\lambda_{1uv} = 0$. Hence for all $s = 0, \dots, n-i-1$, we have that $F_{suv} = \nu_{suv} x_u x_v$. It follows that $\nu_{suv} = 0$. In particular $\nu_{fuv} = \nu_{guv} = \nu_{1uv} = 0$. Then by (19) we have that $\nu_{fvj} = (-1)^{v-u} \nu_{fuj}$, $\nu_{gvj} = (-1)^{v-u} \nu_{guj}$, and $\nu_{1uj} = \nu_{1vj} = 0$. As above, $\nu_{1uj} = 0$ implies that $\nu_{fuj} = (-1)^u \lambda_{1f}^j$ and $\nu_{guj} = (-1)^u \lambda_{1g}^j$. It follows that $F_{fgv} = (-1)^{v-j+1} x_v G_{fg}$, and so $x_v G_{fg} \in I$.

We proceed by induction on l . Suppose that $l \geq 3$, and that $x_z = x_{u_l}$ is introduced inductively from $x_u = x_{u_1}$. Since $x_{u_p} G_{fg} \in I$ for all $1 \leq p < l$, by Lemma 5.2 we may assume that $\mu_{1u_p u_l} = 0$ for all $1 \leq p < l$.

Let $d < e < f$. Observe that if $x_d x_f \in I$ and $x_e x_f \notin I$, then $\nu_{def} = 0$. If $x_d x_f \in I$ and $x_d x_e \notin I$, then $\lambda_{def} = 0$. It follows that $\lambda_{1u_1 u_p} = 0$ for all $1 < p \leq l$, and $\lambda_{u_{p-1} u_p u_m} = 0$ for all $1 < p < l$ and $p < m \leq l$. By inductively applying (16), we have that $\lambda_{1u_{p-1} u_p} = 0$ for all $1 < p \leq l$.

Similarly, applying (18), it follows that $\mu_{1u_{p-1} u_p} = 0$ for all $1 < p \leq l$. Since $x_{u_{p-1}} x_{u_p} \notin I$, we have that $\nu_{su_{p-1} u_p} = 0$ for all $1 < p \leq l$ and $0 \leq s \leq n-i-1$. Then by (19) we have that for all $1 < p \leq l$ and $0 \leq s \leq n-i-1$, $\nu_{su_{p-1} u_p} = (-1)^{u_p - u_{p-1}} \nu_{su_{p-1} j}$. It follows that $\nu_{fzj} = (-1)^{z-u} \nu_{fuj}$, $\nu_{gzj} = (-1)^{z-u} \nu_{guj}$, and $\nu_{1uj} = \nu_{1zj} = 0$. Hence $\nu_{fuj} = (-1)^u \lambda_{1f}^j$ and $\nu_{guj} = (-1)^u \lambda_{1g}^j$. It follows that $F_{fgz} = (-1)^{z-j+1} x_z G_{fg}$, and so $x_z G_{fg} \in I$.

Now suppose that $x_u L_{2q}^j \notin I$. As in the proof of Lemma 3.6 we assume that $f < g < q < u < j$. Recall that q has the property that the coefficient of x_q in G_{fg} and in G_{gq} is zero; that is $\mu_{fq}^j = (-1)^q \lambda_{1f}^j$, and $\mu_{gq}^j = (-1)^q \lambda_{1g}^j$.

Let $x_u \in V_1$. As before we may assume that $\nu_{quj} = 0$, and so $\lambda_{fq}^u = \lambda_{gq}^u = 0$. Then $F_{fqu} = F_{gqu} = 0$, since $x_q x_u \notin I$. It follows that $\nu_{fuj} = (-1)^{u-q} \mu_{fq}^j = (-1)^u \lambda_{1f}^j$ and $\nu_{guj} = (-1)^u \lambda_{1g}^j$. Then $F_{fgu} = (-1)^{u-j+1} x_u G_{fg}$, and so $x_u G_{fg} \in I$.

If $l \geq 2$ we repeat the proof of the previous case, with x_q instead of x_1 , and we obtain that $x_z G_{fg} \in I$.

Now we only need to consider the case when (20) is given by (15),

$$(B_1, B_{d_2}, \dots, B_{d_r}, D_{j_1})(V_N, V_L \setminus V_{D_{j_1}}, B_{d_{r+1}}, \dots, B_{d_k}) \subset I.$$

Notice that (15) includes the cases (9), (12), and (13). Here the linear forms $h_1, \dots, h_{n-i-1-t}$ are all monomials.

We will need the following observations.

Remark 5.8. Let $u \in \{n-i, \dots, n\}$ be such that $x_u L_{ab}^j \notin I$ for some $L_{ab}^j \in \{L_1, \dots, L_t\}$. Let $x_p \in \{h_1, \dots, h_{n-i-1-t}\}$, and assume that $x_a x_p \in I$, $x_b x_p \in I$, that L_{ap}^j is a monomial in x_a , and L_{bp}^j is a monomial in x_b . Suppose that $x_z \in V_l$, $l \geq 1$, is introduced inductively from $x_u \in V_1$. Then $x_z x_p \in I$, by Lemma 5.7.

Lemma 5.9. Let $u \in \{n-i, \dots, n\}$ be such that $x_u L_{ab}^j \notin I$ for some $L_{ab}^j \in \{L_1, \dots, L_t\}$. Let $x_p \in \{h_1, \dots, h_{n-i-1-t}\}$, and assume that $L_{ap}^j = L_{bp}^j = 0$. Suppose that $x_z \in V_l$, $l \geq 1$, is introduced inductively from $x_u \in V_1$. Then $F_{pzj} \in (x_p)$, and $x_z x_p \in I$.

Proof. Assume that $a < b < p < z < j$. Let $x_z \in V_l$, $l \geq 1$. By Remark 5.5 we have that $\lambda_{ap}^z = \pm \nu_{pzj}$, $\lambda_{bp}^z = \pm \nu_{pzj}$, $\mu_{ap}^z = \pm \nu_{azj}$, and $\mu_{bp}^z = \pm \nu_{bzj}$. We show by induction on l that $\nu_{pzj} = 0$ and that $x_p x_z \in I$.

If $l = 1$, then $z = u$, and $\nu_{puj} = 0$ by Lemma 5.6. It follows that $\lambda_{ap}^u = \lambda_{bp}^u = 0$, and so $F_{apu} = \mu_{ap}^u x_u x_p$, and $F_{bpu} = \mu_{bp}^u x_u x_p$. Since $x_j L_{ab}^j \in I$ and $x_u L_{ab}^j \notin I$, by Corollary 5.3 we have that $(\nu_{auj}, \nu_{buj}) \neq (0, 0)$. Therefore $(\mu_{ap}^u, \mu_{bp}^u) \neq (0, 0)$ and $x_p x_u \in I$.

Now suppose that $l > 1$ and that $x_z \in V_l$ is introduced because $x_w x_z \notin I$ for some $x_w \in V_{l-1}$; that is, $x_z H_{c_{l-1}w} \notin I$ for some $H_{c_{l-1}w} \in H_{V_{l-1}}$. Assuming $c_{l-1} < w < z < j$, by construction we have that $(\nu_{c_{l-1}zj}, \nu_{wzj}) \neq (0, 0)$.

Now $x_p x_w \in I$, and $x_w x_z \notin I$ imply that $\nu_{pwz} = 0$. Since $\nu_{pwj} = 0$ by the induction hypothesis, it follows from (19) that $\nu_{pzj} = 0$. Then $\lambda_{ap}^z = \lambda_{bp}^z = 0$, and so $F_{apz} = \mu_{ap}^z x_z x_p$, and $F_{bpz} = \mu_{bp}^z x_z x_p$.

If $c_{l-1} = c_1 \in \{a, b\}$ and $\nu_{c_1 zj} \neq 0$, then $(\mu_{ap}^z, \mu_{bp}^z) \neq (0, 0)$ and $x_z x_p \in I$. Otherwise $\nu_{c_1 zj} \neq 0$, where $x_{c_1} \in V_p$ for some $p < l$, so that $x_{c_1} x_p \in I$. If $x_z x_p \notin I$, then we have that $\mu_{pc_1 z} = \mu_{c_1 c_1 z} = 0$. Recall that we are assuming $\mu_{c_1 c_l j} = 0$. Then by (18) we have that $\nu_{c_l zj} = 0$, a contradiction. \square

Lemma 5.10. Let $u \in \{n-i, \dots, n\}$ and $\{p, q\} \subset \{0, \dots, n-i-1\}$. Suppose that $x_u x_p \in I$, that $F_{puj} \in (x_p)$, and that the coefficient of x_q in L_{pq}^j is not zero. Then $x_u x_q \in I$.

Proof. Assume $p < q < u < j$. By Remark 5.5 we have that $\mu_{pq}^u = \pm \mu_{pq}^j \neq 0$. Then $F_{pqu} = x_u (\lambda_{pq}^u x_p + \mu_{pq}^u x_q)$ and $x_u x_p \in I$ imply that $x_u x_q \in I$. \square

Lemma 5.11. Let $u \in \{n-i, \dots, n\}$ be such that $x_u L_{ab}^j \notin I$ for some $L_{ab}^j \in \{L_1, \dots, L_t\}$. Let $x_p \in \{h_1, \dots, h_{n-i-1-t}\}$, and assume that L_{ap}^j and L_{bp}^j are monomials in x_p . Suppose that $x_z x_p \in I$ for all x_z introduced inductively from $x_u \in V_1$. If the coefficient of x_q in L_{pq}^j is not zero, then $x_z x_q \in I$.

Proof. By Lemma 5.6 we have that $\nu_{puj} = 0$. Now the proof of Lemma 5.9 shows that $\nu_{pzj} = 0$. Then by Lemma 5.10 we have that $x_z x_q \in I$. \square

We are now ready to conclude the proof that $(V_l)(h_1, \dots, h_{n-i-1-t}) \subset I$ for all $l \geq 1$. Let $u \in \{n-i, \dots, n\}$ be such that $x_u L_{ab}^j \notin I$ for some $L_{ab}^j \in \{L_1, \dots, L_t\}$. Suppose that $x_z \in V_l$, $l \geq 1$, is introduced inductively from $x_u \in V_1$. Let $x_p \in \{V_N, V_L \setminus V_{D_{j_1}}, B_{d_{r+1}}, \dots, B_{d_k}\}$. Let $x_{a_{d_{r+1}}}, \dots, x_{a_{d_k}}$ be generators of $B_{d_{r+1}}, \dots, B_{d_k}$ respectively.

First assume that $L_{ab}^j \in D_{j_1}$. If $x_p \in V_N \cup (V_L \setminus V_{D_{j_1}}) \cup \{x_{a_{d_{r+1}}}\} \cup \dots \cup \{x_{a_{d_k}}\}$, then $x_z x_p \in I$ by Lemma 5.9, since $L_{ap}^j = L_{bp}^j = 0$. If $x_p \in \{B_{d_{r+1}}, \dots, B_{d_k}\}$ is not a generator of one of the blocks, then $x_z x_p \in I$ by inductively applying Lemma 5.11.

Next assume that L_{ab}^j is one of the generators of $B_1, B_{d_2}, \dots, B_{d_r}$. Recall that $x_b \in V_{D_{j_1}}$. If $x_p \in V_N \cup (V_L \setminus V_{D_{j_1}})$, we may assume that $L_{ap}^j = 0$, otherwise by Lemma 3.6 and by the proof of Lemma 3.14, we can reduce to the previous case given by equation (7). If $x_p \in \{x_{a_{d_{r+1}}}\} \cup \dots \cup \{x_{a_{d_k}}\}$ we have that $L_{ap}^j = 0$ by Remark 3.9. We also have that $L_{bp}^j = 0$ if $x_p \in V_N \cup (V_L \setminus V_{D_{j_1}}) \cup \{x_{a_{d_{r+1}}}\} \cup \dots \cup \{x_{a_{d_k}}\}$. Then by Lemma 5.9 we have that $x_z x_p \in I$.

If $x_p \in \{B_{d_{r+1}}, \dots, B_{d_k}\}$ is not a generator, then L_{ap}^j is a monomial in x_p (otherwise the block containing x_a would not be maximal), and L_{bp}^j is a monomial in x_p . Then $x_z x_p \in I$ by inductively applying Lemma 5.11.

Last assume that x_a and x_b belong to one of the blocks $B_1, B_{d_2}, \dots, B_{d_r}$. If $x_p \in V_N \cup (V_L \setminus V_{D_{j_1}}) \cup \{x_{a_{d_{r+1}}}\} \cup \dots \cup \{x_{a_{d_k}}\}$, then L_{ap}^j and L_{bp}^j are monomials in x_a and x_b respectively, and so by Remark 5.8 we have that $x_z x_p \in I$. The same holds if x_p is not a generator of $B_{d_{r+1}}, \dots, B_{d_k}$, but the block containing x_p has been constructed after the block containing x_a and x_b .

If the block containing x_p has been constructed before the block containing x_a and x_b , then L_{ap}^j and L_{bp}^j are monomials in x_p and by inductively applying Lemma 5.11 we have that $x_z x_p \in I$.

This concludes the proof of $(V_l)(h_1, \dots, h_{n-i-1-t}) \subset I$, of Theorem 4.1, and of Conjecture 1.1. □

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